SMOOTH STRUCTURES ON ESCHENBURG SPACES: NUMERICAL COMPUTATIONS

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Abstract. This paper numerically computes the topological and smooth invariants of Eschenburg-Kruggel spaces with small fourth cohomology group, following Kruggel's determination of the Kreck-Stolz invariants of special Eschenburg spaces [5, 10, 7]. The Gnu Gmp arbitrary-precision library is utilised [6].

1. Introduction

In [1], Aloff and Wallach introduced a family of 7-manifolds that are homogeneous spaces of SU_3 as follows: let p, q be coprime integers and let $U_{p,q} \subset \mathrm{SU}_3$ be the subgroup of diagonal matrices of the form $\mathrm{diag}(z^p, z^q, z^{-p-q})$ for $z \in S^1$. The Aloff-Wallach 7-manifold $M_{pq} = \mathrm{SU}_3/U_{p,q}$. Aloff and Wallach showed that a bi-invariant metric on SU_3 induced a positively-curved submersion metric on the quotient $M_{p,q}$. In [7, 8], Kreck and Stolz studied the topological and smooth classification of Aloff-Wallach spaces. Amongst other things, they showed that there are diffeomorphic Aloff-Wallach spaces that are not SU_3 equivariantly diffeomorphic: the 'smallest' example occurs with (p, q) equal to $(-4\,638\,661, 582\,656)$ and $(-2\,594\,149, 5\,052\,965)$! [8, p. 468] Each of these spaces has a finite cyclic fourth integral cohomology group; they showed, through a computer search, that if the order of $H^4(M_{p,q}; \mathbf{Z})$ is less than $r = 2\,955\,27\,597$, then the topological structure determines the smooth structure. Additional computer searches, attributed to Zagier and Odlyzko, revealed homeomorphic, but not diffeomorphic, Aloff-Wallach spaces with the rank of $H^4(M_{p,q}; \mathbf{Z})$ between the above number and roughly 2×10^{20} [8, p. 467]. In all cases, there were no reported examples of a topological Aloff-Wallach space whose 28 distinct smooth structures are themselves diffeomorphic to Aloff-Wallach spaces.

In [5], Eschenburg introduced a family of 7-manifolds that generalise Aloff-Wallach spaces [10]. Let $U \cong \mathrm{U}_1$ be a subgroup of $\mathrm{U}_3 \times \mathrm{U}_3$ such that the natural action of U on U_3 defined by

$$\forall u = (u_1, u_2) \in U, g \in \mathrm{U}_3 : \qquad u \cdot g = u_1 g u_2^{-1} \tag{1}$$

stabilises SU_3 and is free. U is conjugate to a diagonal subgroup characterised by 2 integer vectors k and $l \in \mathbf{Z}^3$ such that $k_0 + k_1 + k_2 = l_0 + l_1 + l_2$:

$$U_{kl} = \left\{ \mathrm{diag}(z^{k_0}, z^{k_1}, z^{k_2}) \oplus \mathrm{diag}(z^{l_0}, z^{l_1}, z^{l_2}) \ : \ z \in S^1 \right\}. \tag{2}$$

The freeness of the action (1) is equivalent to the property that

$$\forall \text{ permutations } \sigma : \qquad k - \sigma(l) \text{ is a primitive vector in } \mathbf{Z}^3. \tag{3}$$

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Eschenburg defined k, l to be *admissible* if

$$\begin{array}{ll} \gcd(k_0 - l_0, k_1 - l_1), & \gcd(k_0 - l_0, k_1 - l_2), \\ \gcd(k_0 - l_1, k_1 - l_0), & \gcd(k_0 - l_1, k_1 - l_2), \\ \gcd(k_0 - l_2, k_1 - l_0), & \text{and } \gcd(k_0 - l_2, k_1 - l_1) \quad \text{equal } 1. \end{array} \tag{4}$$

Definition 1. *Let $k, l \in \mathbf{Z}^3$ satisfy $k_0 + k_1 + k_2 = l_0 + l_1 + l_2$ and the admissibility conditions* (4) *and define U_{kl} as in* (2)*. The 7-manifold $E_{kl} := \mathrm{SU}_3/U_{kl}$ is called an* Eschenburg *space.*

Eschenburg computed the integral cohomology ring of $E_{k,l} = \mathrm{SU}_3/U_{k,l}$ and proved that these spaces were strongly inhomogeneous in most cases. He also showed that under certain conditions on k, l, a bi-invariant metric on SU_3 induces a positively-curved submersion metric on $E_{k,l}$.

In [10], Kruggel computed the Kreck-Stolz invariants of a broad number of Eschenburg spaces—henceforth an Eschenburg-Kruggel space—and obtained a classification of these Eschenburg-Kruggel spaces up to homotopy, homeomorphism and diffeomorphism. In [4], Chinburg, Escher and Ziller implemented a computer search for homeomorphic, but not diffeomorphic, positively-curved (resp. 3-Sasakian) Eschenburg-Kruggel spaces. They found that for $\#H^4(E_{k,l}; \mathbf{Z}) < 8000$, there is a unique pair of homeomorphic, but not diffeomorphic, positively-curved Eschenburg-Kruggel spaces. In [3], the present author proved that the existence of a real-analytically completely integrable convex Hamiltonian is a non-trivial smooth invariant of the configuration space, and proved the complete integrability of geodesic flows on all Eschenburg-Kruggel spaces. That work motivated the

Question 1. *Let E be a topological Eschenburg space. Is each smooth structure on E diffeomorphic to an Eschenburg space $E_{k,l}$?*

One knows from the work of Kreck and Stolz that each topological Eschenburg space admits 28 distinct oriented smooth structures, but one does not know if each structure is represented by an Eschenburg space. From the above-mentioned results, it is not clear whether each distinct oriented smooth structure on a topological Eschenburg space is represented by an Eschenburg space or if such representatives are rather sparse, as for Aloff-Wallach spaces. This note attempts to cast some light on this question.

Theorem 1. *Let $I = [-850, 850]$ and $J = [1, 101]$. Amongst the Eschenburg-Kruggel spaces with $(k, l) \in I^3 \times I^3$, for each odd $|r| = \#H^4(E; \mathbf{Z})$ in the interval J, columns 2 & 9 of Table 1 show a lower bound on the number of oriented homeomorphism classes. For $|r| \leq 9$, each oriented homeomorphism class of Eschenburg-Kruggel spaces has each of its* 28 *distinct oriented smooth structures represented by an Eschenburg-Kruggel space $E_{k,l}$ with $(k, l) \in I^3 \times I^3$.*

Remark 1. Columns 3–7 & 10–14 of Table 1 list the number of topological Eschenburg-Kruggel spaces, for a fixed $|r|$, that have the stated number of oriented smooth structures represented by Eschenburg-Kruggel spaces.

The smooth structures on a topological Eschenburg-Kruggel space are an orbit of the group of homotopy 7-spheres ($\cong \mathbf{Z}_{28}$). The Kreck-Stolz invariant s_1 is additive under this action: if Σ is a homotopy 7-sphere and E is an Eschenburg-Kruggel space, then $s_1(E\#\Sigma) = s_1(E) + s_1(\Sigma)$ and $28 \cdot s_1(\Sigma) \equiv 0 \bmod 1$. This implies that each topological Eschenburg-Kruggel space has 28 distinct oriented smooth structures [10]. The difficulty is that the surgery description of the smooth structure $E\#\Sigma$ does not appear to contain information about the structure of $E\#\Sigma$ as a Eschenburg-Kruggel space.

Tables 5–7 list representative Eschenburg-Kruggel spaces for each smooth structure on each topological Eschenburg-Kruggel space with $|r| \leq 5$ that was found in

constructing table 1. It seems likely that all topological and smooth Eschenburg-Kruggel spaces with $|r| \leq 5$ are enumerated in tables 5–7.

Table 1: $|r| = \operatorname{rank} H^4(E; \mathbf{Z})$ versus the number of homeomorphism classes (#Top.), and the number of homeomorphism classes with the n smooth structures represented by Eschenburg-Kruggel spaces, for $n = 28, 27$, $14 \leq n \leq 26$, $2 \leq n \leq 13$ and $n = 1$.

		Counts							Counts				
$\lvert r\rvert$	$\#Top.$	28	27	14-26	2-13	1	$\lvert r\rvert$	$\#Top.$	28	27	14-26	2-13	1
1	12	12					3	8	8				
5	48	48					7	120	120				
9	24	24					11	360	354	4	2		
13	576	542	22	12			15	32	32				
17	1152	988	68	96			19	1512	1216	86	204	6	
21	80	64	10	6			23	2640	1726	276	598	40	
25	240	240					27	72	72				
29	4704	1656	814	2212	22		31	5760	1506	794	3080	380	
33	240	114	30	92	4		35	480	230	90	160		
37	8634	904	918	5728	1072	12	39	384	118	58	176	32	
41	11988	376	636	8778	2176	22	43	12600	272	500	9412	2414	2
45	96	60	20	16			47	17108	82	248	10950	5812	16
49	1848	1028	310	510			51	768	44	26	522	176	
53	22456	46	122	11414	10836	38	55	1440	320	200	752	168	
57	1008	28	36	666	278		59	29902	10	32	10662	19034	164
61	32874	22	76	9468	22764	544	63	240	60	10	164	6	
65	2304	178	220	1332	574		67	39854	12	28	7890	31108	816
69	1756			864	878	14	71	47544			6596	39738	1210
73	48034	2	10	6090	40864	1068	75	160	50	42	68		
77	3600	332	112	1914	1234	8	79	59046		4	4508	51962	2572
81	216	188	22	6			83	67340			3544	59816	3980
85	4602	28	82	2670	1800	22	87	3128			580	2522	26
89	78944			2068	70090	6786	91	5740	256	154	2502	2734	94
93	3788			468	3182	138	95	6016	18	22	2836	3054	86
97	91772			1484	79690	10598	99	720	12	40	474	194	
101	100490			742	87290	12458							

This note is structured as follows: section 2 reviews Kruggel's condition C; section 3 reviews Kruggel's computation of the Kreck-Stolz invariants; section 4 explains how the Kreck-Stolz invariants were computed in software; and appendices 5.1–5.4 add several tables.

2. Eschenburg-Kruggel spaces

To compute the Kreck-Stolz invariants of Eschenburg spaces, Kruggel observed that the projection of an $x \in \mathrm{SU}_3$ onto its first two columns in the Steifel manifold $\mathrm{V}_2(\mathbf{C}^3)$ is a diffeomorphism. From the embedding of $\mathrm{V}_2(\mathbf{C}^3) \subset \mathbf{C}^{2\cdot 3}$, Kruggel constructed an 8-manifold W' with boundary $\mathrm{V}_2(\mathbf{C}^3)$. The action of U_{kl} descends naturally to $\mathbf{C}^{2\cdot 3}$ and W', but the action on W' has 3 singular orbits. One can cut away these three singular orbits to construct a cobordism between E_{kl} and a union of 3 lens spaces – provided that the matrix

$$A = \begin{bmatrix} k_0 - l_0 & k_0 - l_1 & k_0 - l_2 \\ k_1 - l_0 & k_1 - l_1 & k_1 - l_2 \\ k_2 - l_0 & k_2 - l_1 & k_2 - l_2 \end{bmatrix} \tag{5}$$

has a column or row containing non-zero, pairwise coprime entries.

Definition 2 (Kruggel 2006)**.** *The Eschenburg space $E_{k,l}$ satisfies* condition C *iff the matrix A has a column or row containing non-zero, pairwise coprime entries. An Eschenburg space that satisfies condition C is called an* Eschenburg-Kruggel space.

Remark 2. Note that the coprimality conditions (4) do not imply that all entries of A are non-zero. The Eschenburg space E_{kl} with $k = (-1, -1, 2)$ and $l = (-2, 0, 2)$

has

$$A = \begin{bmatrix} 1 & -1 & -3 \\ 1 & -1 & -3 \\ 4 & 2 & 0 \end{bmatrix}. \tag{6}$$

This defines an Eschenburg-Kruggel space according to definition 2. Indeed, the coprimality conditions (4) are satisfied, since they are

$$\begin{array}{llcll} \gcd(A_{00}, A_{11}), & \gcd(A_{00}, A_{12}), & & \gcd(1,-1), & \gcd(1,-3), \\ \gcd(A_{01}, A_{10}), & \gcd(A_{01}, A_{12}), & \textit{i.e.} & \gcd(-1,1), & \gcd(-1,-3), \\ \gcd(A_{02}, A_{10}), & \gcd(A_{02}, A_{11}) & & \gcd(-3,1), & \text{and } \gcd(-3,-1) \end{array} \tag{7}$$

which are all unity; and condition C is satisfied by the left-most column of A. See remark 3 for more.

3. Invariants of Eschenburg-Kruggel spaces

Let E_{kl} be an Eschenburg space. Let u be the Chern class of the bundle $S^1 = U_{kl} \hookrightarrow \mathrm{SU}_3 \to E_{kl}$. Eschenburg proved that the non-trivial parts of the integral cohomology ring of $E_{k,l}$ has the following structure:

$$H^2(E_{kl}; \mathbf{Z}) = \mathbf{Z} \cdot u, \qquad H^4(E_{kl}; \mathbf{Z}) = \mathbf{Z}_r \cdot u^2. \tag{8}$$

The integer $r = \sigma_2(k) - \sigma_2(l)$ where σ_j is the j-th elementary symmetric polynomial, $\sigma_j(x) = \sum_{i_1 < \cdots < i_j} x_{i_1} \cdots x_{i_j}$. The linking form of $E_{k,l}$ is plainly determined by the linking number of u^2 with itself. Kruggel showed that this equals

$$\mathrm{Lk}(u^2, u^2) = -\frac{s^{-1}}{r} \bmod 1, \tag{9}$$

where $s = \sigma_3(k) - \sigma_3(l)$ and s^{-1} is the multiplicative inverse of $s \bmod r$. Kruggel also showed that the first Pontryagin class of E_{kl} equals

$$p_1(E_{kl}) = p_1 \cdot u^2 \bmod r \qquad \text{where } p_1 = 2\sigma_1(k)^2 - 6\sigma_2(k). \tag{10}$$

Although this expression appears to be asymmetric in k and l, the sum condition plus the definition of r ensures that it is well-defined.

In addition to the above invariants, Kruggel was able to compute the Kreck-Stolz invariants for Eschenburg-Kruggel spaces. To explain, let $p \neq 0$ be coprime to the non-zero integers $p_0, \ldots, p_3$, and let

$$\begin{aligned} &L = L(p; p_0, p_1, p_2, p_3) = S^7/C \text{ where} \\ &C = \{\mathrm{diag}(e^{\frac{2\pi i k p_0}{p}}, e^{\frac{2\pi i k p_1}{p}}, e^{\frac{2\pi i k p_2}{p}}, e^{\frac{2\pi i k p_3}{p}}) \;:\; k = 0, \ldots, p-1\} \end{aligned} \tag{11}$$

be a lens space. Define the following functions

$$s_1(L) = \frac{1}{2^7 \cdot 7 \cdot p} \sum_{k=1}^{|p|-1} \prod_{j=0}^{3} \cot\left(\frac{k\pi p_j}{p}\right) + \frac{1}{2^4 \cdot p} \sum_{k=1}^{|p|-1} \prod_{j=0}^{3} \csc\left(\frac{k\pi p_j}{p}\right) \quad \bmod 1 \tag{12}$$

$$s_2(L) = \frac{1}{2^4 \cdot p} \sum_{k=1}^{|p|-1} \left(e^{\frac{2\pi i k}{p}} - 1\right) \prod_{j=0}^{3} \csc\left(\frac{k\pi p_j}{p}\right) \quad \bmod 1$$

These are the Kreck-Stolz invariants of the lens space L in (11), and they take values in $\mathbf{Q}/\mathbf{Z}$.

Assume that the left-most column of the matrix A has pairwise coprime, non-zero entries (the remaining cases are described below). The above-described cobordism exhibits E_{kl} as cobordant to the disjoint union of the three lens spaces:

$$\begin{aligned} &L_0 = L(A_{00}; A_{10}, A_{20}, A_{11}, A_{21}) \qquad L_1 = L(A_{10}; A_{00}, A_{20}, A_{01}, A_{21}) \\ &L_2 = L(A_{20}; A_{00}, A_{10}, A_{01}, A_{11}). \end{aligned} \tag{13}$$

Let us see that L_0 is indeed a lens space. By condition C, the integers A_{j0} are pairwise coprime and non-zero. The primitivity condition (3) implies that A_{00} is coprime to A_{11} and A_{21}. For example, suppose that A_{00} and A_{11} have a divisor $d > 1$, so one can write $A_{00} + A_{11} = dc$. The condition that $\sum k_i = \sum l_i$ in definition 1 is equivalent to $A_{00} + A_{11} + A_{22} = 0$, so $A_{22} = -dc$. If $c = 0$, then the vector $k - l = (A_{00}, A_{11}, A_{22})$ is not primitive; if $c \neq 0$, then the same vector is not primitive, too. (This argument also shows that if $A_{22} = 0$, then $A_{00} = -A_{11} = \pm 1$.) The remaining verifications for L_1 and L_2 are similar.

Kruggel showed that the Kreck-Stolz invariants are equal to

$$s_1(E_{kl}) = \frac{\mathrm{sign}(w)}{2^5 \cdot 7} - \frac{q^2}{2^7 \cdot 7 \cdot w} - \sum_{i=1}^{3} s_1(L_i) \qquad \mathrm{mod}\ 1 \qquad (14)$$

$$s_2(E_{kl}) = \frac{q-2}{2^4 \cdot 3 \cdot w} - \sum_{i=1}^{3} s_2(L_i) \qquad \mathrm{mod}\ 1 \qquad (15)$$

where

$$q = A_{00}^2 + A_{10}^2 + A_{20}^2 + A_{01}^2 + A_{11}^2 + A_{21}^2 - (l_0 - l_1)^2, \qquad (16)$$

$$w = r \cdot A_{00} A_{10} A_{20}. \qquad (17)$$

These invariants are transcendental functions of the variables k, l. This fact, plus the fact that the sums can have a rather large number of terms, means that showing two Eschenburg-Kruggel spaces are homeomorphic or diffeomorphic is rather difficult. However, since s_i is a rational integer, one can use a few numerical tricks to prove equality of these invariants.

Remark 3 (*c.f.* remark 2)**.** The well-definedness of Kruggel's formulae (14–15) amounts to the statement that if $E_{k,l}$ satisfies condition C, then w (17) does not vanish. Indeed, from the remark above, the lens-space invariants $s_j(L_i)$ (12–13) are well-defined if $w \neq 0$. Since condition C is assumed to hold for the left-most column of A, $A_{00}A_{10}A_{20} \neq 0$. In addition, Kruggel proved that r must be odd [10, p. 572] (in fact, since $H^3(E_{k,l}; \mathbf{Z})$ vanishes, Poincaré duality implies $r \neq 0$). Therefore, $w \neq 0$.

The results of this note rely on

Theorem 2 (Kruggel 2005)**.** *Two Eschenburg-Kruggel spaces, $E_{k,l}$ and $E_{k',l'}$ are orientation-preserving* homeomorphic *if $|r|, s, p_1$ and s_2 coincide. If, in addition, s_1 coincides, then they are orientation-preserving* diffeomorphic*.*

3.1. **Automorphisms and invariants.** To compute the Kreck-Stolz invariants of Eschenburg-Kruggel spaces in general, one uses the extension of the natural action of Weyl group of $\mathrm{SU}_3 \times \mathrm{SU}_3$ by the automorphism that interchanges factors. Concretely, let S_3 be the symmetric group acting naturally on $\mathbf{Z}^3$ by permutations, let τ be the involutive automorphism of $\mathbf{Z}^3 \oplus \mathbf{Z}^3$ which acts by $(k, l) \mapsto (l, k)$ and let $\eta : (k, l) \mapsto (-k, -l)$.

The group generated by $S_3 \times S_3$, τ (resp. $S_3 \times S_3$, τ and η) is denoted by $\mathfrak{G}^+$ (resp. $\mathfrak{G}$). $\mathfrak{G}$ is a group of order 144 and $\mathfrak{G}^+$ is an index 2 subgroup.

Proposition 1 ([5])**.** *For each $\sigma \in \mathfrak{G}$, the Eschenburg spaces $E_{k,l}$ and $E_{\sigma(k,l)}$ are diffeomorphic. If $\sigma \in \mathfrak{G}^+$, they are orientation-preserving diffeomorphic.*

Remark 4. With the above proposition, the formulae for the Kreck-Stolz invariants can be extended to all Eschenburg-Kruggel spaces as follows. The Eschenburg space E_{kl} is orientation-preserving diffeomorphic to $E_{\alpha(k),\beta(l)}$ for any permutations $\alpha, \beta \in S_3$. In addition, $E_{k,l}$ is orientation-preserving diffeomorphic to $E_{l,k}$. The

permutation α permutes the rows (resp. β permutes the columns) of A, while the diffeomorphism $E_{kl} \to E_{lk}$ induces $A \mapsto -A'$.

It follows that if the column j (resp. row j) of A has non-zero pairwise coprime entries, then the leftmost column of $A_{k,\beta(l)}$ (resp. $A_{l,\beta(k)}$) has non-zero pairwise coprime entries and E_{kl} is orientation-preserving diffeomorphic to $E_{k,\beta(l)}$ (resp. $E_{l,\beta(k)}$) where $\beta = (0\, j)$. By this observation, one can compute the Kreck-Stolz invariants of any Eschenburg-Kruggel space by means of the formulae (14,15).

The proposition also implies that each Eschenburg space $E_{k,l}$ has a representative, up to orientation, where $k_0 \leq k_1 \leq k_2$, $l_0 \leq l_1 \leq l_2$ and $k_0 \leq l_0$.

4. Methodology

The search for homeomorphic smooth Eschenburg-Kruggel space neatly divides into three separate searches:

(1) search over a domain of parameters $(k, l) \in \mathbf{Z}^3 \times \mathbf{Z}^3$ for Eschenburg-Kruggel spaces;
(2) computation of the invariants r, s, p_1 and s_1, s_2 in terms of the parameters (k, l);
(3) search the data generated for matching invariants.

Due to the size of the sample space considered, it was decided to do the first two steps in compiled code. The structure of the problem led to the choice of `C++` as the language of choice.

The computations to generate all of the tables in this note took approximately six weeks of continuous cpu time on a single core of a 2-core 3.0GHz Intel Core Duo E6850 cpu with 4MB cache and 3.3GB DDRAM 4.0GB swap. The operating system was RHEL with the 2.6.8 Linux kernel.

4.1. **The search over parameter space.**
Let us define the parameter space and explain how the search is conducted.

4.1.1. *The parameter space.*
Let $\mathbf{1} \in \mathbf{Z}^3$ be the vector whose elements are all unity. If $E_{k,l}$ is an Eschenburg space, then $E_{k+n\mathbf{1},l+n\mathbf{1}}$ is the same Eschenburg space for any $n \in \mathbf{Z}$. There is, therefore, a unique representative of $(k, l) + \mathbf{Z}(\mathbf{1}, \mathbf{1})$ such that $\sum k_i = \sum l_i \in [0, 2]$. All searches were conducted with this constraint.[1]

4.1.2. *The search.*
The speed of the arithmetic in the native `signed long int` class of integers in `C++` argued in favour of performing testing the admissibility condition (4) and condition C (definition 2) in `signed long int`.

The coprimality tests are conducted by a two-part process. First, an $N \times N$ lookup table is created. The (i, j) entry of the lookup table equals 1 if i and j are coprime and $ij \neq 0$; otherwise, it is 0. If $|i|$ or $|j|$ exceed N, the Euclidean algorithm is first employed to reduce both i and j until the lookup table can be used. The parameter N is chosen at compile time; in our tests $N = 2000$ was chosen so that all coprimality tests required only a lookup.

4.2. **Computation of the invariants.**
This is broken into two parts.

[1]In tables 5–7, one finds the sums reported lie in $[-2, 2]$. Those spaces with sum reported in $[-2, -1]$ are obtained by reversing the orientation of a space whose sum lies in $[1, 2]$.

4.2.1. *Integer invariants.* If (k,l) define an Eschenburg-Kruggel space, then the rank of $H^4(E;\mathbf{Z})$, $|r|$, and the first Pontryagin class p_1 were computed using `signed long int` arithmetic. Since the set of `signed long ints` equals $[-2^{31}, 2^{31}-1] \cap \mathbf{Z}$, and both r and p_1 are quadratic forms in (k,l), `signed long int` arithmetic does not run into under/overflow errors for $|k_i| < 10922$. For the purposes of this note, all computations of r and p_1 were done in `signed long int` arithmetic.

Since s is cubic in (k,l), under/overflow does not affect computation for $|k_i|, |l_i| < 1023$. This relatively small bound led us to use GMP arbitrary precision floats to compute s (see below).

4.2.2. *Rational invariants.* From the definition of the Kreck-Stolz invariants (12), one can see that individual terms in each summand can be $O(1/p^3)$.

The GNU GMP package, along with its GMPFRXX front-end for `C++`, permit one to do arbitrary precision arithmetic from within `C++` [6, 14]. Since GNU GMP can compute the trigonometric functions to arbitrary precision, we elected to use this package to compute the Kreck-Stolz invariants of an Eschenburg-Kruggel space.

The relative slowness of software-implemented arithmetic also indicated a need to permit computation with machine-native floating point arithmetic. The template facility of `C++` made it possible to use the same code for both machine-native and software-implemented floating-point arithmetic and allow the user to choose the precision at run-time rather than compile-time.

4.3. **Matching invariants.** The final step was to match the topological and smooth invariants that are computed for different Eschenburg-Kruggel spaces. This was accomplished, in essence, by multiple sorts. In the first step, a `C++` programme computed and sorted approximately 2GB of the polynomial Eschenburg-Kruggel space invariants (r, s, p_1). These data were stored in text files, and these were sorted and split according to the value of $|r|$. The Kreck-Stolz invariants of these spaces were computed with 130 bits of precision and stored in a second database. The resulting data were imported into a second `C++` programme where homeomorphism and diffeomorphism classes were computed. The data for tables 1–7 were generated in this way.

4.3.1. *Testing.* To ensure the accuracy of the computations, several tests were designed. These included:

(1) replication of each of the published computations in [2, section 4], table 1 of [10] and tables 1-6 of [4][2] and table 1 of [10];
(2) replication, up to a numerical $\epsilon \sim 2^{-130}$, of closed-form answers for the invariants of some Eschenburg-Kruggel space;
(3) replication, up to a numerical $\epsilon \sim 2^{-130}$, of the `C++` computed results in MAPLE, MAXIMA and BC [11, 12, 13].

5. APPENDICES

5.1. **Appendix A.** The graph in figure 1 graphs the number N of Eschenburg-Kruggel spaces in the cube $[-k,k]^6$, as a function of k, with the constraint that $\sum k_i = \sum l_i \in [0,2]$. A rough heuristic indicates that $N = O(k^4)$ for large k and $\Delta N = O(k^3)$–which is nicely captured here. It is also apparent that $\Delta N(k)$ grows like $c_\pm k^3$, where $c_\pm$ depends only on the parity of k.

[2]In replicating these results, differing conventions for the projection map $x \mapsto \bar{x} \in (-\frac{1}{2}, \frac{1}{2}]$ became apparent. The Chinburg-Escher-Ziller code uses the convention that x is reduced mod 1, then $[0, \frac{1}{2}]$ is mapped to itself by the identity and $(\frac{1}{2}, 1]$ is mapped to $(-1, 0]$ by a constant shift. In our `C++` code, x is reduced mod 1, then shifted by $-\frac{1}{2}$.

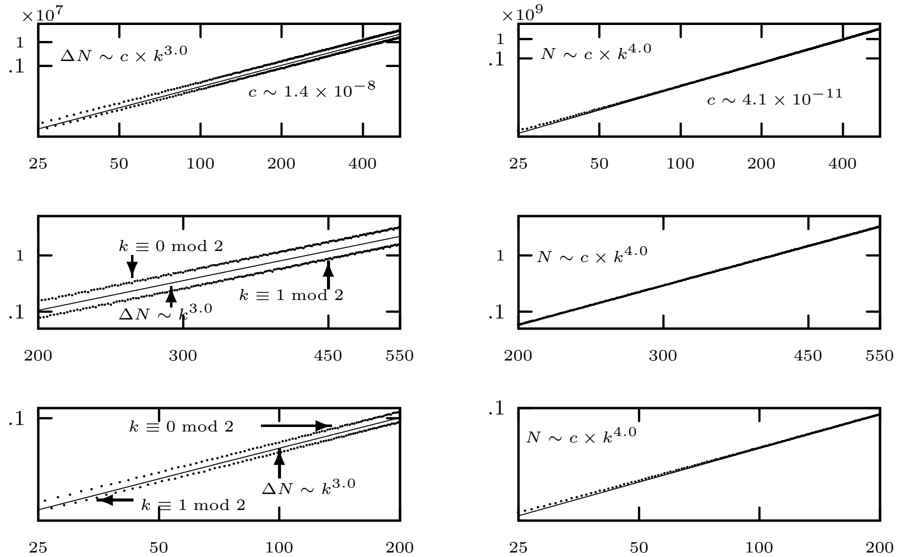

Figure 1: [Log-Log scale]. The number of Eschenburg-Kruggel spaces, $N = N(k)$, and the marginal number, $\Delta N = \Delta N(k)$, in the cube $[-k, k]^6$. Left column, descending: the marginal number for k in the intervals $[25, 550]$, $[200, 550]$ and $[25, 200]$; Right column, descending: the total number for the same intervals. A least-squares regression line is also displayed on each graph.

5.2. **Appendix B.** We observed several unexplained phenomena. For fixed invariants r, s and p_1, the Kreck-Stolz invariant s_2 appears to lie in the orbit of $\mathbf{Z}_n$ acting by $x \mapsto x + \frac{1}{n}$ mod 1 where $n = 4$ or 12. We also observed that the values taken on by s_1 appear to depend only on $|r|$, s and p_1.

The first columns of Tables 2 and 3 shows these group actions on the Kreck-Stolz invariants, when $|r| = 1, 3$. Table 4 abstracts the picture from tables 2 and 3, and shows the group actions on s_2 and s_1. It appears that $\mathbf{Z}_{12}$ acts effectively except when $r \equiv 0 \bmod 3$, $r \not\equiv 0 \bmod 3^2$, in which case $\mathbf{Z}_4$ acts effectively.

Additional tables are available at http://www.maths.ed.ac.uk/~lbutler/kspace/.

Table 4: A summary of the invariants of Eschenburg-Kruggel spaces, where $k \in \mathbf{Z}_{12}$, $l \in \mathbf{Z}_{28}$ and s_1, s_2 take values in $\left[-\frac{1}{2}, \frac{1}{2}\right)$ mod 1.

$\lvert r\rvert$	s	p_1	s_2	s_1	$\lvert r\rvert$	s	p_1	s_2	s_1
1	0	0	$0/1 + k/12$	$-1/2 + l/28$					
3	−1	0	$2/9 + k/4$	$-55/112 + l/28$	3	1	0	$1/36 + k/4$	$-53/112 + l/28$
5	−2	2	$0/1 + k/12$	$-69/140 + l/28$	5	−1	−2	$1/60 + k/12$	$-131/280 + l/28$
5	1	−2	$1/15 + k/12$	$-139/280 + l/28$	5	2	2	$0/1 + k/12$	$-33/70 + l/28$
7	−3	−3	$1/84 + k/12$	$-389/784 + l/28$	7	−3	1	$1/28 + k/12$	$-373/784 + l/28$
7	−3	2	$0/1 + k/12$	$-365/784 + l/28$	7	−2	0	$1/42 + k/12$	$-53/112 + l/28$
7	−1	0	$1/21 + k/12$	$-53/112 + l/28$	7	1	0	$1/28 + k/12$	$-55/112 + l/28$
7	2	0	$5/84 + k/12$	$-55/112 + l/28$	7	3	−3	$1/14 + k/12$	$-367/784 + l/28$
7	3	1	$1/21 + k/12$	$-383/784 + l/28$	7	3	2	$0/1 + k/12$	$-391/784 + l/28$
9	−4	0	$2/27 + k/12$	$-1/2 + l/28$	9	4	0	$1/108 + k/12$	$-1/2 + l/28$
11	−5	−5	$1/66 + k/12$	$-83/176 + l/28$	11	−5	−1	$1/33 + k/12$	$-597/1232 + l/28$
11	−5	4	$1/132 + k/12$	$-87/176 + l/28$	11	−4	−4	$5/66 + k/12$	$-577/1232 + l/28$
11	−4	2	$0/1 + k/12$	$-87/176 + l/28$	11	−4	5	$1/22 + k/12$	$-597/1232 + l/28$
11	−3	−1	$1/44 + k/12$	$-577/1232 + l/28$	11	−3	2	$0/1 + k/12$	$-83/176 + l/28$
11	−3	3	$5/66 + k/12$	$-573/1232 + l/28$	11	−2	−4	$3/44 + k/12$	$-615/1232 + l/28$
11	−2	−3	$1/66 + k/12$	$-579/1232 + l/28$	11	−2	1	$7/132 + k/12$	$-607/1232 + l/28$
11	−1	−5	$5/66 + k/12$	$-573/1232 + l/28$	11	−1	−3	$1/33 + k/12$	$-597/1232 + l/28$
11	−1	−2	$1/132 + k/12$	$-577/1232 + l/28$	11	1	−5	$1/132 + k/12$	$-615/1232 + l/28$

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Table 4, continued from previous page

$\lvert r\rvert$	s	p_1	s_2	s_1	$\lvert r\rvert$	s	p_1	s_2	s_1
11	1	−3	$7/132+k/12$	$-591/1232+l/28$	11	1	−2	$5/66+k/12$	$-611/1232+l/28$
11	2	−4	$1/66+k/12$	$-573/1232+l/28$	11	2	−3	$3/44+k/12$	$-87/176+l/28$
11	2	1	$1/33+k/12$	$-83/176+l/28$	11	3	−1	$2/33+k/12$	$-611/1232+l/28$
11	3	2	$0/1+k/12$	$-607/1232+l/28$	11	3	3	$1/132+k/12$	$-615/1232+l/28$
11	4	−4	$1/132+k/12$	$-611/1232+l/28$	11	4	2	$0/1+k/12$	$-579/1232+l/28$
11	4	5	$5/132+k/12$	$-591/1232+l/28$	11	5	−5	$3/44+k/12$	$-607/1232+l/28$
11	5	−1	$7/132+k/12$	$-591/1232+l/28$	11	5	4	$5/66+k/12$	$-579/1232+l/28$
13	−6	−6	$3/52+k/12$	$-181/364+l/28$	13	−6	−5	$1/52+k/12$	$-173/364+l/28$
13	−6	−2	$11/156+k/12$	$-179/364+l/28$	13	−6	0	$1/13+k/12$	$-13/28+l/28$
13	−5	−3	$5/78+k/12$	$-85/182+l/28$	13	−5	−1	$1/26+k/12$	$-43/91+l/28$
13	−5	0	$1/39+k/12$	$-1/2+l/28$	13	−5	4	$3/52+k/12$	$-89/182+l/28$
13	−4	0	$5/156+k/12$	$-27/56+l/28$	13	−4	2	$0/1+k/12$	$-355/728+l/28$
13	−4	5	$1/13+k/12$	$-363/728+l/28$	13	−4	6	$1/52+k/12$	$-361/728+l/28$
13	−3	−6	$5/156+k/12$	$-347/728+l/28$	13	−3	−5	$1/26+k/12$	$-341/728+l/28$
13	−3	−2	$3/52+k/12$	$-339/728+l/28$	13	−3	0	$11/156+k/12$	$-27/56+l/28$
13	−2	0	$5/78+k/12$	$-1/2+l/28$	13	−2	2	$0/1+k/12$	$-173/364+l/28$
13	−2	5	$11/156+k/12$	$-181/364+l/28$	13	−2	6	$1/26+k/12$	$-179/364+l/28$
13	−1	−4	$2/39+k/12$	$-363/728+l/28$	13	−1	0	$7/156+k/12$	$-27/56+l/28$
13	−1	1	$5/78+k/12$	$-355/728+l/28$	13	−1	3	$1/52+k/12$	$-361/728+l/28$
13	1	−4	$5/156+k/12$	$-339/728+l/28$	13	1	0	$1/26+k/12$	$-27/56+l/28$
13	1	1	$1/52+k/12$	$-347/728+l/28$	13	1	3	$5/78+k/12$	$-341/728+l/28$
13	2	0	$1/52+k/12$	$-1/2+l/28$	13	2	2	$0/1+k/12$	$-89/182+l/28$
13	2	5	$1/78+k/12$	$-85/182+l/28$	13	2	6	$7/156+k/12$	$-43/91+l/28$
13	3	−6	$2/39+k/12$	$-355/728+l/28$	13	3	−5	$7/156+k/12$	$-361/728+l/28$
13	3	−2	$1/39+k/12$	$-363/728+l/28$	13	3	0	$1/78+k/12$	$-27/56+l/28$
13	4	0	$2/39+k/12$	$-27/56+l/28$	13	4	2	$0/1+k/12$	$-347/728+l/28$
13	4	5	$1/156+k/12$	$-339/728+l/28$	13	4	6	$5/78+k/12$	$-341/728+l/28$
13	5	−3	$1/52+k/12$	$-181/364+l/28$	13	5	−1	$7/156+k/12$	$-179/364+l/28$
13	5	0	$3/52+k/12$	$-13/28+l/28$	13	5	4	$1/39+k/12$	$-173/364+l/28$
13	6	−6	$1/39+k/12$	$-85/182+l/28$	13	6	−5	$5/78+k/12$	$-89/182+l/28$
13	6	−2	$1/78+k/12$	$-43/91+l/28$	13	6	0	$1/156+k/12$	$-1/2+l/28$
15	−7	−3	$7/36+k/4$	$-267/560+l/28$	15	−4	3	$11/45+k/4$	$-261/560+l/28$
15	−2	−3	$1/18+k/4$	$-277/560+l/28$	15	−1	3	$13/90+k/4$	$-269/560+l/28$
15	1	3	$19/180+k/4$	$-271/560+l/28$	15	2	−3	$7/36+k/4$	$-263/560+l/28$
15	4	3	$1/180+k/4$	$-279/560+l/28$	15	7	−3	$1/18+k/4$	$-39/80+l/28$

5.3. **Appendix C.** Tables 5–7 list homeomorphism classes of Eschenburg-Kruggel spaces with the rank of the fourth integral cohomology group equal to $|r| = 1, 3, 5$ respectively. Each smooth structure in each such homeomorphism class is represented by an Eschenburg-Kruggel space; these tables list the 'smallest' representatives.

Table 5: Smallest Eschenburg-Kruggel spaces with $|r| = 1$, $s_2 \geq 0$.

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $\lvert r\rvert = 1$, $s = 0$, $p_1 = 0$, $s_2 = 0/1$					
sum	k_0	k_1	l_0	l_1	s_1
0	−17	5	−14	−2	$-1/2$
0	−24	6	−13	−12	$-13/28$
0	−47	11	−38	−8	$-3/7$
0	−7	3	−6	0	$-11/28$
0	−17	5	−16	2	$-5/14$
0	−36	18	−29	−4	$-9/28$
0	−27	−1	−24	−6	$-2/7$
0	7	5	6	6	$-1/4$
0	−41	−3	−38	−8	$-3/14$
0	−12	−1	−10	−4	$-5/28$
0	−17	−4	−14	−8	$-1/7$
0	−11	−9	−10	−10	$-3/28$
0	−24	6	−23	3	$-1/14$
0	3	1	2	2	$-1/28$
0	−4	1	−2	−2	$0/1$
0	−3	−1	−2	−2	$1/28$
0	−20	−3	−18	−6	$1/14$
0	11	9	10	10	$3/28$
0	−22	8	−21	4	$1/7$
0	−14	4	−13	1	$5/28$
0	−46	8	−44	3	$3/14$
0	−7	−5	−6	−6	$1/4$
0	−30	6	−28	1	$2/7$
0	−33	4	−18	−18	$9/28$
0	−14	−2	−12	−5	$5/14$
0	−6	0	−4	−3	$11/28$
0	−46	8	−36	−11	$3/7$
0	−25	12	−18	−6	$13/28$

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $\lvert r\rvert = 1$, $s = 0$, $p_1 = 0$, $s_2 = 1/12$					
sum	k_0	k_1	l_0	l_1	s_1
1	−26	6	−23	−1	$-1/2$
−2	10	8	9	9	$-13/28$
1	−14	5	−7	−7	$-3/7$
1	−21	−3	−19	−6	$-11/28$
1	−11	4	−9	−1	$-5/14$
1	−29	9	−22	−6	$-9/28$
1	−38	2	−33	−7	$-2/7$
1	−30	10	−19	−11	$-1/4$
1	−13	−1	−11	−4	$-3/14$
1	−6	2	−3	−3	$-5/28$
1	−14	−3	−11	−7	$-1/7$
1	−13	5	−10	−2	$-3/28$
1	−34	−2	−25	−15	$-1/14$
1	−17	−1	−12	−8	$-1/28$
1	−43	19	−42	13	$0/1$
1	−17	−5	−12	−11	$1/28$
1	−11	−2	−7	−7	$1/14$
1	−10	5	−9	1	$3/28$
1	−27	8	−21	−5	$1/7$
1	−42	14	−27	−15	$5/28$
1	−21	11	−20	5	$3/14$
1	−3	−1	−2	−2	$1/4$
1	−17	5	−11	−6	$2/7$
1	−43	5	−34	−11	$9/28$
1	−20	0	−17	−5	$5/14$
−2	2	0	1	1	$11/28$
1	−5	3	−4	0	$3/7$
1	−13	7	−11	0	$13/28$

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Table 5, continued from previous page

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $\lvert r\rvert = 1$, $s = 0$, $p_1 = 0$, $s_2 = 1/6$					
sum	k_0	k_1	l_0	l_1	s_1
−1	26	−8	17	9	−13/28
−1	30	−11	17	13	−3/7
−1	17	−7	16	−3	−11/28
−1	14	12	13	13	−5/14
−1	49	−7	40	10	−9/28
−1	26	24	25	25	−2/7
−1	13	−7	12	−2	−1/4
−1	21	−4	13	9	−3/14
−1	17	−3	10	8	−5/28
−1	33	−7	29	2	−1/7
−1	32	−14	29	−3	−3/28
−1	21	−2	17	5	−1/14
−1	46	−11	41	1	−1/28
−1	2	0	1	1	0/1
−1	24	−3	13	13	1/28
−1	21	−7	13	8	1/14
−1	10	−3	5	5	3/28
2	−15	−13	−14	−14	1/7
−1	9	−3	5	4	5/28
2	−3	−1	−2	−2	3/14
−1	30	−11	29	−7	1/4
2	−7	−5	−6	−6	2/7
−1	29	1	18	16	9/28
−1	6	4	5	5	5/14
−1	45	5	29	26	11/28
−1	14	−4	9	5	3/7
−1	52	5	41	21	13/28
2	−11	−9	−10	−10	1/2

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $\lvert r\rvert = 1$, $s = 0$, $p_1 = 0$, $s_2 = 1/4$					
sum	k_0	k_1	l_0	l_1	s_1
0	−22	−6	−19	−10	−1/2
0	−50	1	−38	−18	−13/28
0	−48	−4	−45	−9	−3/7
0	−20	2	−11	−11	−11/28
0	−9	3	−8	0	−5/14
0	−22	6	−18	−3	−9/28
0	−19	5	−16	−2	−2/7
0	2	0	1	1	−1/4
0	−39	6	−22	−20	−3/14
0	−62	30	−59	13	−5/28
0	−20	0	−17	−5	−1/7
0	−4	−2	−3	−3	−3/28
0	−48	22	−35	−11	−1/14
0	−20	0	−13	−10	−1/28
0	−28	10	−27	6	0/1
0	6	4	5	5	1/28
0	−14	6	−11	−2	1/14
0	−19	5	−18	2	3/28
0	−15	6	−14	2	1/7
0	−49	11	−40	−8	5/28
0	−41	19	−28	−12	3/14
0	−26	12	−23	1	1/4
0	−28	6	−26	1	2/7
0	−48	−10	−31	−31	9/28
0	−48	16	−45	6	5/14
0	−43	14	−42	10	11/28
0	−10	−1	−8	−4	3/7
0	10	8	9	9	13/28

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $\lvert r\rvert = 1$, $s = 0$, $p_1 = 0$, $s_2 = 1/3$					
sum	k_0	k_1	l_0	l_1	s_1
1	−19	7	−16	−1	−1/2
1	−19	9	−18	4	−13/28
1	−17	8	−15	1	−3/7
−2	1	−1	0	0	−11/28
1	−8	2	−5	−3	−5/14
1	−38	8	−33	−3	−9/28
1	−47	−3	−40	−14	−2/7
1	−3	1	−2	−1	−1/4
1	−14	0	−9	−7	−3/14
1	−17	9	−14	−1	−5/28
1	−11	−1	−9	−4	−1/7
1	−10	−1	−7	−5	−3/28
1	−24	−1	−17	−11	−1/14
1	−29	13	−25	0	−1/28
1	−20	−2	−15	−9	0/1
1	−5	1	−4	−1	1/28
1	−26	−4	−21	−11	1/14
1	−15	5	−12	−2	3/28
1	−30	8	−29	5	1/7
1	−15	5	−9	−6	5/28
1	−7	3	−6	0	3/14
1	−4	−2	−3	−3	1/4
1	−9	0	−7	−3	2/7
1	−9	4	−7	−1	9/28
1	−10	4	−9	1	5/14
1	−17	6	−13	−3	11/28
1	−23	7	−18	−4	3/7
1	−7	1	−6	−1	13/28

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $\lvert r\rvert = 1$, $s = 0$, $p_1 = 0$, $s_2 = 5/12$					
sum	k_0	k_1	l_0	l_1	s_1
−1	23	−5	12	12	−13/28
−1	44	−17	25	19	−3/7
−1	31	−7	20	12	−11/28
−1	7	0	5	3	−5/14
−1	31	−16	25	3	−9/28
−1	9	−5	8	−1	−2/7
−1	10	−1	9	1	−1/4
−1	23	−7	15	8	−3/14
−1	17	−9	12	4	−5/28
−1	25	−11	22	−1	−1/7
−1	6	−1	5	1	−3/28
−1	42	−17	21	21	−1/14
−1	5	3	4	4	−1/28
−1	24	−9	23	−5	0/1
−1	14	−1	13	1	1/28
−1	35	−15	28	4	1/14
−1	15	−8	11	3	3/28
−1	15	−1	12	4	1/7
−1	30	−1	17	17	5/28
−1	15	−5	8	7	3/14
−1	4	−1	3	1	1/4
−1	17	−5	15	0	2/7
−1	32	−16	27	1	9/28
−1	11	−3	7	4	5/14
−1	8	−1	7	1	11/28
−1	12	−4	7	5	3/7
−1	2	−1	1	1	13/28
−1	24	7	21	11	1/2

Computed Kreck-Stolz invariants of Eschenburg-Kruggel spaces														
s_2	s_1													
$-1/2$	$-1/2$	$-13/28$	$-3/7$	$-11/28$	$-5/14$	$-9/28$	$-2/7$	$-1/4$	$-3/14$	$-5/28$	$-1/7$	$-3/28$	$-1/14$	$-1/28$
	$0/1$	$1/28$	$1/14$	$3/28$	$1/7$	$5/28$	$3/14$	$1/4$	$2/7$	$9/28$	$5/14$	$11/28$	$3/7$	$13/28$
$-5/12$	$-1/2$	$-13/28$	$-3/7$	$-11/28$	$-5/14$	$-9/28$	$-2/7$	$-1/4$	$-3/14$	$-5/28$	$-1/7$	$-3/28$	$-1/14$	$-1/28$
	$0/1$	$1/28$	$1/14$	$3/28$	$1/7$	$5/28$	$3/14$	$1/4$	$2/7$	$9/28$	$5/14$	$11/28$	$3/7$	$13/28$
$-1/3$	$-1/2$	$-13/28$	$-3/7$	$-11/28$	$-5/14$	$-9/28$	$-2/7$	$-1/4$	$-3/14$	$-5/28$	$-1/7$	$-3/28$	$-1/14$	$-1/28$
	$0/1$	$1/28$	$1/14$	$3/28$	$1/7$	$5/28$	$3/14$	$1/4$	$2/7$	$9/28$	$5/14$	$11/28$	$3/7$	$13/28$
$-1/4$	$-1/2$	$-13/28$	$-3/7$	$-11/28$	$-5/14$	$-9/28$	$-2/7$	$-1/4$	$-3/14$	$-5/28$	$-1/7$	$-3/28$	$-1/14$	$-1/28$
	$0/1$	$1/28$	$1/14$	$3/28$	$1/7$	$5/28$	$3/14$	$1/4$	$2/7$	$9/28$	$5/14$	$11/28$	$3/7$	$13/28$
$-1/6$	$-1/2$	$-13/28$	$-3/7$	$-11/28$	$-5/14$	$-9/28$	$-2/7$	$-1/4$	$-3/14$	$-5/28$	$-1/7$	$-3/28$	$-1/14$	$-1/28$
	$0/1$	$1/28$	$1/14$	$3/28$	$1/7$	$5/28$	$3/14$	$1/4$	$2/7$	$9/28$	$5/14$	$11/28$	$3/7$	$13/28$
$-1/12$	$-1/2$	$-13/28$	$-3/7$	$-11/28$	$-5/14$	$-9/28$	$-2/7$	$-1/4$	$-3/14$	$-5/28$	$-1/7$	$-3/28$	$-1/14$	$-1/28$
	$0/1$	$1/28$	$1/14$	$3/28$	$1/7$	$5/28$	$3/14$	$1/4$	$2/7$	$9/28$	$5/14$	$11/28$	$3/7$	$13/28$
$0/1$	$-1/2$	$-13/28$	$-3/7$	$-11/28$	$-5/14$	$-9/28$	$-2/7$	$-1/4$	$-3/14$	$-5/28$	$-1/7$	$-3/28$	$-1/14$	$-1/28$
	$0/1$	$1/28$	$1/14$	$3/28$	$1/7$	$5/28$	$3/14$	$1/4$	$2/7$	$9/28$	$5/14$	$11/28$	$3/7$	$13/28$
$1/12$	$-13/28$	$-3/7$	$-11/28$	$-5/14$	$-9/28$	$-2/7$	$-1/4$	$-3/14$	$-5/28$	$-1/7$	$-3/28$	$-1/14$	$-1/28$	$0/1$
	$1/28$	$1/14$	$3/28$	$1/7$	$5/28$	$3/14$	$1/4$	$2/7$	$9/28$	$5/14$	$11/28$	$3/7$	$13/28$	$1/2$
$1/6$	$-13/28$	$-3/7$	$-11/28$	$-5/14$	$-9/28$	$-2/7$	$-1/4$	$-3/14$	$-5/28$	$-1/7$	$-3/28$	$-1/14$	$-1/28$	$0/1$
	$1/28$	$1/14$	$3/28$	$1/7$	$5/28$	$3/14$	$1/4$	$2/7$	$9/28$	$5/14$	$11/28$	$3/7$	$13/28$	$1/2$
$1/4$	$-13/28$	$-3/7$	$-11/28$	$-5/14$	$-9/28$	$-2/7$	$-1/4$	$-3/14$	$-5/28$	$-1/7$	$-3/28$	$-1/14$	$-1/28$	$0/1$
	$1/28$	$1/14$	$3/28$	$1/7$	$5/28$	$3/14$	$1/4$	$2/7$	$9/28$	$5/14$	$11/28$	$3/7$	$13/28$	$1/2$
$1/3$	$-13/28$	$-3/7$	$-11/28$	$-5/14$	$-9/28$	$-2/7$	$-1/4$	$-3/14$	$-5/28$	$-1/7$	$-3/28$	$-1/14$	$-1/28$	$0/1$
	$1/28$	$1/14$	$3/28$	$1/7$	$5/28$	$3/14$	$1/4$	$2/7$	$9/28$	$5/14$	$11/28$	$3/7$	$13/28$	$1/2$
$5/12$	$-13/28$	$-3/7$	$-11/28$	$-5/14$	$-9/28$	$-2/7$	$-1/4$	$-3/14$	$-5/28$	$-1/7$	$-3/28$	$-1/14$	$-1/28$	$0/1$
	$1/28$	$1/14$	$3/28$	$1/7$	$5/28$	$3/14$	$1/4$	$2/7$	$9/28$	$5/14$	$11/28$	$3/7$	$13/28$	$1/2$

Table 2: $|r| = 1$, $s \equiv 0, p_1 \equiv 0 \bmod r$.

Computed Kreck-Stolz invariants of Eschenburg-Kruggel spaces with $|r| = 3$, $s \equiv -1, p_1 \equiv 0 \bmod r$.

s_2	s_1													
$-5/18$	$-55/112$	$-51/112$	$-47/112$	$-43/112$	$-39/112$	$-5/16$	$-31/112$	$-27/112$	$-23/112$	$-19/112$	$-15/112$	$-11/112$	$-1/16$	$-3/112$
	$1/112$	$5/112$	$9/112$	$13/112$	$17/112$	$3/16$	$25/112$	$29/112$	$33/112$	$37/112$	$41/112$	$45/112$	$7/16$	$53/112$
$-1/36$	$-55/112$	$-51/112$	$-47/112$	$-43/112$	$-39/112$	$-5/16$	$-31/112$	$-27/112$	$-23/112$	$-19/112$	$-15/112$	$-11/112$	$-1/16$	$-3/112$
	$1/112$	$5/112$	$9/112$	$13/112$	$17/112$	$3/16$	$25/112$	$29/112$	$33/112$	$37/112$	$41/112$	$45/112$	$7/16$	$53/112$
$2/9$	$-55/112$	$-51/112$	$-47/112$	$-43/112$	$-39/112$	$-5/16$	$-31/112$	$-27/112$	$-23/112$	$-19/112$	$-15/112$	$-11/112$	$-1/16$	$-3/112$
	$1/112$	$5/112$	$9/112$	$13/112$	$17/112$	$3/16$	$25/112$	$29/112$	$33/112$	$37/112$	$41/112$	$45/112$	$7/16$	$53/112$
$17/36$	$-55/112$	$-51/112$	$-47/112$	$-43/112$	$-39/112$	$-5/16$	$-31/112$	$-27/112$	$-23/112$	$-19/112$	$-15/112$	$-11/112$	$-1/16$	$-3/112$
	$1/112$	$5/112$	$9/112$	$13/112$	$17/112$	$3/16$	$25/112$	$29/112$	$33/112$	$37/112$	$41/112$	$45/112$	$7/16$	$53/112$

Computed Kreck-Stolz invariants of Eschenburg-Kruggel spaces with $|r| = 3$, $s \equiv 1, p_1 \equiv 0 \bmod r$.

s_2	s_1													
$-17/36$	$-53/112$	$-7/16$	$-45/112$	$-41/112$	$-37/112$	$-33/112$	$-29/112$	$-25/112$	$-3/16$	$-17/112$	$-13/112$	$-9/112$	$-5/112$	$-1/112$
	$3/112$	$1/16$	$11/112$	$15/112$	$19/112$	$23/112$	$27/112$	$31/112$	$5/16$	$39/112$	$43/112$	$47/112$	$51/112$	$55/112$
$-2/9$	$-53/112$	$-7/16$	$-45/112$	$-41/112$	$-37/112$	$-33/112$	$-29/112$	$-25/112$	$-3/16$	$-17/112$	$-13/112$	$-9/112$	$-5/112$	$-1/112$
	$3/112$	$1/16$	$11/112$	$15/112$	$19/112$	$23/112$	$27/112$	$31/112$	$5/16$	$39/112$	$43/112$	$47/112$	$51/112$	$55/112$
$1/36$	$-53/112$	$-7/16$	$-45/112$	$-41/112$	$-37/112$	$-33/112$	$-29/112$	$-25/112$	$-3/16$	$-17/112$	$-13/112$	$-9/112$	$-5/112$	$-1/112$
	$3/112$	$1/16$	$11/112$	$15/112$	$19/112$	$23/112$	$27/112$	$31/112$	$5/16$	$39/112$	$43/112$	$47/112$	$51/112$	$55/112$
$5/18$	$-53/112$	$-7/16$	$-45/112$	$-41/112$	$-37/112$	$-33/112$	$-29/112$	$-25/112$	$-3/16$	$-17/112$	$-13/112$	$-9/112$	$-5/112$	$-1/112$
	$3/112$	$1/16$	$11/112$	$15/112$	$19/112$	$23/112$	$27/112$	$31/112$	$5/16$	$39/112$	$43/112$	$47/112$	$51/112$	$55/112$

Table 3: $|r| = 3$, $s \equiv \pm 1, p_1 \equiv 0 \bmod r$.

Table 6: Smallest Eschenburg-Kruggel spaces with $|r| = 3$, $s_2 \geq 0$.

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $\lvert r\rvert = 3$, $s = -1$, $p_1 = 0$, $s_2 = 2/9$					
sum	k_0	k_1	l_0	l_1	s_1
0	-165	60	-152	16	-55/112
0	-129	45	-128	40	-51/112
0	-63	-36	-56	-44	-47/112
0	-51	15	-32	-20	-43/112
0	-132	27	-128	16	-39/112
0	-164	-20	-156	-33	-5/16
0	-36	-15	-32	-20	-31/112
0	-68	16	-60	-3	-27/112
0	-39	12	-20	-20	-23/112
0	-156	45	-92	-68	-19/112
0	-104	-20	-81	-51	-15/112
0	-164	52	-108	-57	-11/112
0	-132	51	-104	-20	-1/16
0	-81	36	-80	28	-3/112
0	-12	-3	-8	-8	1/112
0	-60	-27	-56	-32	5/112
0	-177	36	-164	4	9/112
0	-60	15	-56	4	13/112
0	-57	21	-56	16	17/112
0	-92	40	-75	-9	3/16
0	-84	9	-68	-20	25/112
0	-56	16	-36	-21	29/112
0	-75	-9	-68	-20	33/112
0	-33	-3	-20	-20	37/112
0	-21	9	-20	4	41/112
0	-15	3	-8	-8	45/112
0	-84	33	-80	16	7/16
0	-129	36	-104	-20	53/112

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $\lvert r\rvert = 3$, $s = -1$, $p_1 = 0$, $s_2 = 17/36$					
sum	k_0	k_1	l_0	l_1	s_1
0	-2	1	0	0	-55/112
0	-66	12	-41	-29	-51/112
0	-89	19	-84	6	-47/112
0	-29	7	-18	-12	-43/112
0	-35	-11	-24	-24	-39/112
0	-50	7	-30	-24	-5/16
0	-6	0	-5	-2	-31/112
0	-42	12	-26	-17	-27/112
0	-30	12	-29	7	-23/112
0	-54	24	-50	7	-19/112
0	-30	0	-29	-2	-15/112
0	-174	60	-170	43	-11/112
0	-24	0	-23	-2	-1/16
0	-66	6	-53	-17	-3/112
0	-42	0	-41	-2	1/112
0	-54	18	-29	-26	5/112
0	-41	-2	-30	-18	9/112
0	-36	-18	-29	-26	13/112
0	-90	42	-77	-2	17/112
0	-114	-12	-74	-65	3/16
0	-36	0	-35	-2	25/112
0	-30	-12	-26	-17	29/112
0	-72	6	-50	-29	33/112
0	-48	18	-41	-2	37/112
0	-12	0	-11	-2	41/112
0	-65	19	-42	-24	45/112
0	-18	0	-17	-2	7/16
0	-114	30	-65	-53	53/112

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $\lvert r\rvert = 3$, $s = 1$, $p_1 = 0$, $s_2 = 1/36$					
sum	k_0	k_1	l_0	l_1	s_1
0	-54	0	-37	-25	-53/112
0	-36	6	-25	-13	-7/16
0	-130	47	-114	0	-45/112
0	-54	6	-34	-25	-41/112
0	-10	-1	-6	-6	-37/112
0	-78	30	-73	11	-33/112
0	-37	-1	-30	-12	-29/112
0	-30	-6	-25	-13	-25/112
0	-60	-12	-43	-34	-3/16
0	-34	11	-30	0	-17/112
0	-36	18	-25	-10	-13/112
0	-42	0	-34	-13	-9/112
0	-106	35	-54	-54	-5/112
0	-84	24	-82	17	-1/112
0	-82	29	-72	0	3/112
0	-97	35	-66	-30	1/16
0	-61	23	-42	-18	11/112
0	-120	36	-103	-7	15/112
0	-126	24	-121	11	19/112
0	-25	11	-18	-6	23/112
0	-48	-6	-34	-25	27/112
0	-96	42	-61	-34	31/112
0	-82	-1	-72	-18	5/16
0	-114	-36	-97	-58	39/112
0	-60	18	-58	11	43/112
0	-12	6	-10	-1	47/112
0	-36	12	-34	5	51/112
0	-12	0	-7	-7	55/112

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $\lvert r\rvert = 3$, $s = 1$, $p_1 = 0$, $s_2 = 5/18$					
sum	k_0	k_1	l_0	l_1	s_1
0	-76	-34	-72	-39	-53/112
0	-4	2	-3	0	-7/16
0	-22	2	-15	-9	-45/112
0	-52	14	-48	3	-41/112
0	-46	8	-33	-15	-37/112
0	-93	45	-58	-34	-33/112
0	-24	9	-22	2	-29/112
0	-39	15	-28	-10	-25/112
0	-75	27	-52	-22	-3/16
0	-72	33	-70	20	-17/112
0	-34	-16	-27	-24	-13/112
0	-51	24	-34	-16	-9/112
0	-21	-3	-16	-10	-5/112
0	-22	2	-21	0	-1/112
0	-57	-15	-52	-22	3/112
0	-48	15	-46	8	1/16
0	-111	39	-106	20	11/112
0	-10	2	-9	0	15/112
0	-72	27	-64	2	19/112
0	-52	26	-45	0	23/112
0	-87	-9	-70	-34	27/112
0	-16	2	-15	0	31/112
0	-28	-10	-24	-15	5/16
0	-81	-39	-70	-52	39/112
0	-48	9	-34	-16	43/112
0	-70	26	-48	-21	47/112
0	-34	14	-24	-9	51/112
0	-24	3	-16	-10	55/112

Table 7: Smallest Eschenburg-Kruggel spaces with $|r| = 5$, $s_2 \geq 0$.

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = -2$, $p_1 = 2$, $s_2 = 0/1$

sum	k_0	k_1	l_0	l_1	s_1
0	−168	52	−161	28	−69/140
0	−128	12	−92	−47	−16/35
0	−81	−27	−64	−48	−59/140
0	−228	112	−177	−36	−27/70
0	−76	−7	−48	−44	−7/20
0	−87	3	−68	−28	−11/35
0	−124	−48	−121	−52	−39/140
0	−187	−1	−144	−68	−17/70
0	−88	16	−57	−36	−29/140
0	−147	39	−84	−68	−6/35
0	−116	13	−108	−4	−19/140
0	−47	4	−28	−24	−1/10
0	−168	16	−157	−7	−9/140
0	−161	28	−88	−84	−1/35
0	−228	56	−157	−76	1/140
0	−164	72	−156	33	3/70
0	−12	3	−8	−4	11/140
0	−172	83	−168	52	4/35
0	−52	−1	−48	−8	3/20
0	−64	12	−37	−31	13/70
0	−108	32	−107	28	31/140
0	−188	56	−101	−92	9/35
0	−124	−8	−77	−71	41/140
0	−67	23	−64	12	23/70
0	−127	59	−108	−4	51/140
0	−212	103	−204	52	2/5
0	−144	32	−137	13	61/140
0	−24	−8	−21	−12	33/70

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = -2$, $p_1 = 2$, $s_2 = 1/12$

sum	k_0	k_1	l_0	l_1	s_1
1	−69	35	−42	−26	−69/140
1	−96	44	−95	35	−16/35
1	−99	25	−91	4	−59/140
1	−35	15	−32	4	−27/70
1	−86	18	−85	15	−7/20
1	−55	25	−51	8	−11/35
1	−59	−5	−52	−16	−39/140
1	−95	−5	−91	−12	−17/70
1	−27	−6	−25	−9	−29/140
1	−75	5	−51	−32	−6/35
1	−45	21	−32	−11	−19/140
1	−106	53	−89	−5	−1/10
1	−91	24	−79	−5	−9/140
1	−91	18	−65	−29	−1/35
1	−91	44	−55	−35	1/140
1	−36	8	−35	5	3/70
1	−65	11	−46	−22	11/140
1	−75	31	−67	4	4/35
1	−25	1	−16	−12	3/20
1	−109	55	−91	−6	13/70
1	−149	75	−86	−62	31/140
1	−62	−6	−49	−25	9/35
1	−11	−2	−9	−5	41/140
1	−96	13	−75	−25	23/70
1	−32	−11	−29	−15	51/140
1	−16	4	−15	1	2/5
1	−55	15	−51	4	61/140
1	−105	21	−76	−32	33/70

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = -2$, $p_1 = 2$, $s_2 = 1/6$

sum	k_0	k_1	l_0	l_1	s_1
−1	56	2	45	19	−69/140
−1	46	−3	35	15	−16/35
−1	72	26	65	35	−59/140
−1	36	−18	35	−11	−27/70
−1	37	6	35	9	−7/20
−1	82	16	65	39	−11/35
−1	56	−14	55	−11	−39/140
−1	6	−3	5	−1	−17/70
−1	96	−19	65	35	−29/140
−1	95	−25	56	42	−6/35
−1	66	37	59	45	−19/140
−1	26	−3	25	−1	−1/10
−1	42	21	35	29	−9/140
−1	19	−5	16	2	−1/35
−1	77	−34	69	−5	1/140
−1	36	−3	35	−1	3/70
−1	126	−43	89	35	11/140
−1	55	−11	32	26	4/35
−1	21	2	19	5	3/20
−1	25	−1	21	6	13/70
−1	61	−28	55	−5	31/140
−1	21	−8	15	5	9/35
−1	79	−35	76	−19	41/140
−1	141	−44	75	69	23/70
−1	59	−15	46	12	51/140
−1	26	−8	25	−5	2/5
−1	146	−64	129	−5	61/140
−1	16	−3	15	−1	33/70

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = -2$, $p_1 = 2$, $s_2 = 1/4$

sum	k_0	k_1	l_0	l_1	s_1
0	−166	38	−154	7	−69/140
0	−119	17	−76	−52	−16/35
0	−86	28	−53	−34	−59/140
0	−122	4	−113	−13	−27/70
0	−109	22	−72	−42	−7/20
0	−19	−3	−12	−12	−11/35
0	−63	22	−32	−32	−39/140
0	−93	46	−92	34	−17/70
0	−122	−2	−113	−18	−29/140
0	−18	7	−12	−6	−6/35
0	−53	6	−32	−26	−19/140
0	−82	−22	−74	−33	−1/10
0	−82	−16	−73	−29	−9/140
0	−74	27	−72	18	−1/35
0	−209	102	−122	−86	1/140
0	−69	7	−52	−22	3/70
0	−113	27	−106	8	11/140
0	−112	14	−109	7	4/35
0	−98	11	−92	−2	3/20
0	−36	18	−34	7	13/70
0	−72	−6	−53	−33	31/140
0	−26	−2	−18	−13	9/35
0	−76	8	−58	−23	41/140
0	−23	6	−12	−12	23/70
0	−142	38	−74	−73	51/140
0	−102	−42	−89	−58	2/5
0	−49	−18	−46	−22	61/140
0	−49	22	−46	8	33/70

continued next page

Table 7, continued from previous page

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = -2$, $p_1 = 2$, $s_2 = 1/3$

sum	k_0	k_1	l_0	l_1	s_1
1	−105	40	−101	23	−69/140
1	−160	26	−137	−21	−16/35
1	−25	10	−21	−1	−59/140
1	−137	−1	−120	−30	−27/70
1	−137	59	−105	−24	−7/20
1	−60	30	−41	−17	−11/35
1	−20	6	−17	−1	−39/140
1	−61	19	−60	15	−17/70
1	−61	−1	−49	−20	−29/140
1	−61	19	−54	0	−6/35
1	−140	46	−81	−61	−19/140
1	−225	76	−221	59	−1/10
1	−81	39	−54	−25	−9/140
1	−65	20	−57	−1	−1/35
1	−57	−21	−54	−25	1/140
1	−181	79	−164	15	3/70
1	−41	19	−30	−9	11/140
1	−181	79	−180	70	4/35
1	−81	39	−70	0	3/20
1	−49	−10	−41	−21	13/70
1	−70	−25	−57	−41	31/140
1	−121	19	−105	−14	9/35
1	−61	−37	−50	−49	41/140
1	−97	19	−89	0	23/70
1	−89	0	−61	−41	51/140
1	−204	95	−141	−57	2/5
1	−64	10	−61	3	61/140
1	−97	−61	−89	−70	33/70

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = -2$, $p_1 = 2$, $s_2 = 5/12$

sum	k_0	k_1	l_0	l_1	s_1
−1	110	−6	91	27	−69/140
−1	1	1	0	0	−16/35
−1	39	−10	31	7	−59/140
−1	67	−9	54	15	−27/70
−1	81	47	74	55	−7/20
−1	61	−13	34	30	−11/35
−1	41	7	39	10	−39/140
−1	31	−3	30	−1	−17/70
−1	57	11	55	14	−29/140
−1	30	−6	27	1	−6/35
−1	111	7	90	39	−19/140
−1	11	−3	10	−1	−1/10
−1	161	−43	120	40	−9/140
−1	21	−3	20	−1	−1/35
−1	31	−9	30	−6	1/140
−1	71	−3	70	−1	3/70
−1	41	−13	30	10	11/140
−1	97	−9	74	30	4/35
−1	151	−33	150	−30	3/20
−1	21	7	15	14	13/70
−1	51	−13	50	−10	31/140
−1	41	−3	40	−1	9/35
−1	121	−33	90	30	41/140
−1	61	−23	44	15	23/70
−1	71	−33	70	−25	51/140
−1	61	−3	60	−1	2/5
−1	81	−23	60	20	61/140
−1	17	1	15	4	33/70

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = -1$, $p_1 = -2$, $s_2 = 1/60$

sum	k_0	k_1	l_0	l_1	s_1
−1	41	−11	40	−8	−131/280
−1	92	−28	69	21	−121/280
−1	87	−30	49	39	−111/280
−1	19	−9	15	2	−101/280
−1	129	−41	122	−18	−13/40
−1	99	−9	62	47	−81/280
−1	91	−41	90	−33	−71/280
−1	47	−8	29	21	−61/280
−1	22	7	19	11	−51/280
−1	80	−33	69	1	−41/280
−1	55	2	51	9	−31/280
−1	72	−8	49	29	−3/40
−1	191	−41	162	22	−11/280
−1	50	−18	31	19	−1/280
−1	52	15	41	29	9/280
−1	29	−1	22	10	19/280
−1	110	7	89	39	29/280
−1	69	−31	47	20	39/280
−1	49	9	47	12	7/40
−1	90	7	59	49	59/280
−1	55	−18	49	−1	69/280
−1	62	−30	61	−21	79/280
−1	109	−11	72	47	89/280
−1	70	−33	61	−1	99/280
−1	110	2	99	21	109/280
−1	9	−1	7	2	17/40
−1	91	−21	87	−10	129/280
−1	91	−21	90	−18	139/280

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = -1$, $p_1 = -2$, $s_2 = 1/10$

sum	k_0	k_1	l_0	l_1	s_1
0	−24	−9	−18	−16	−131/280
0	−126	−38	−99	−72	−121/280
0	−26	12	−24	3	−111/280
0	−69	21	−58	−6	−101/280
0	−46	4	−39	−9	−13/40
0	−47	16	−46	12	−81/280
0	−118	−6	−104	−29	−71/280
0	−68	14	−67	11	−61/280
0	−97	41	−96	34	−51/280
0	−26	−6	−24	−9	−41/280
0	−18	4	−17	1	−31/280
0	−47	16	−38	−6	−3/40
0	−19	8	−18	4	−11/280
0	−72	21	−66	4	−1/280
0	−144	51	−136	22	9/280
0	−89	16	−58	−36	19/280
0	−79	23	−66	−8	29/280
0	−76	22	−64	−7	39/280
0	−72	−39	−66	−46	7/40
0	−86	32	−79	8	59/280
0	−66	−28	−57	−39	69/280
0	−64	23	−58	4	79/280
0	−32	1	−28	−6	89/280
0	−98	24	−89	1	99/280
0	−29	13	−26	2	109/280
0	−126	14	−89	−47	17/40
0	−156	74	−139	8	129/280
0	−56	2	−39	−24	139/280

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Table 7, continued from previous page

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = -1$, $p_1 = -2$, $s_2 = 11/60$

sum	k_0	k_1	l_0	l_1	s_1
1	−97	13	−94	6	−131/280
1	−7	3	−6	1	−121/280
1	−14	6	−7	−7	−111/280
1	−5	3	−4	1	−101/280
1	−95	3	−74	−31	−13/40
1	−174	81	−147	−7	−81/280
1	−67	−7	−54	−26	−71/280
1	−17	3	−16	1	−61/280
1	−35	13	−34	9	−51/280
1	−79	14	−47	−37	−41/280
1	−47	−25	−39	−34	−31/280
1	−127	23	−114	−6	−3/40
1	−86	1	−77	−15	−11/280
1	−15	3	−14	1	−1/280
1	−106	−54	−95	−67	9/280
1	−37	3	−36	1	19/280
1	−56	−24	−45	−37	29/280
1	−47	3	−46	1	39/280
1	−45	13	−31	−14	7/40
1	−174	86	−107	−65	59/280
1	−115	−7	−111	−14	69/280
1	−27	3	−26	1	79/280
1	−67	−5	−46	−34	89/280
1	−85	43	−64	−16	99/280
1	−79	36	−77	23	109/280
1	−37	−17	−31	−24	17/40
1	−55	23	−39	−14	129/280
1	−57	3	−56	1	139/280

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = -1$, $p_1 = -2$, $s_2 = 4/15$

sum	k_0	k_1	l_0	l_1	s_1
−1	17	5	14	9	−131/280
−1	89	6	77	25	−121/280
−1	124	−31	97	25	−111/280
−1	117	−43	81	34	−101/280
−1	121	−26	117	−15	−13/40
−1	85	−43	84	−31	−81/280
−1	234	−79	197	17	−71/280
−1	5	−3	4	1	−61/280
−1	165	−83	134	16	−51/280
−1	125	−43	81	44	−41/280
−1	37	−15	34	−4	−31/280
−1	117	−23	74	49	−3/40
−1	65	−23	41	24	−11/280
−1	64	−26	57	−3	−1/280
−1	89	−16	57	37	9/280
−1	137	−23	129	−4	19/280
−1	157	−35	134	16	29/280
−1	174	−56	117	57	39/280
−1	66	−31	65	−23	7/40
−1	54	−16	37	17	59/280
−1	65	−3	54	16	69/280
−1	49	−14	45	−3	79/280
−1	114	−36	77	37	89/280
−1	24	9	17	17	99/280
−1	56	14	37	37	109/280
−1	57	−15	46	9	17/40
−1	41	−16	37	−3	129/280
−1	85	−23	74	4	139/280

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = -1$, $p_1 = -2$, $s_2 = 7/20$

sum	k_0	k_1	l_0	l_1	s_1
0	−46	−11	−44	−14	−131/280
0	−126	49	−124	38	−121/280
0	−43	9	−42	6	−111/280
0	−82	26	−71	−3	−101/280
0	−44	16	−38	−1	−13/40
0	−94	38	−86	9	−81/280
0	−143	69	−142	56	−71/280
0	−61	19	−52	−4	−61/280
0	−41	−1	−24	−24	−51/280
0	−14	−2	−11	−6	−41/280
0	−71	2	−62	−14	−31/280
0	−52	26	−51	17	−3/40
0	−72	6	−63	−11	−11/280
0	−86	27	−44	−44	−1/280
0	−82	−4	−51	−46	9/280
0	−111	29	−92	−14	19/280
0	−12	6	−11	2	29/280
0	−6	−1	−4	−4	39/280
0	−114	18	−103	−6	7/40
0	−94	26	−51	−46	59/280
0	−124	38	−111	2	69/280
0	−94	26	−78	−11	79/280
0	−122	−14	−86	−63	89/280
0	−63	−11	−54	−24	99/280
0	−91	−6	−74	−32	109/280
0	−51	17	−44	−2	17/40
0	−53	−21	−44	−32	129/280
0	−104	28	−86	−13	139/280

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = -1$, $p_1 = -2$, $s_2 = 13/30$

sum	k_0	k_1	l_0	l_1	s_1
1	−77	−20	−69	−31	−131/280
1	−2	0	−1	1	−121/280
1	−72	−45	−59	−59	−111/280
1	−70	28	−49	−19	−101/280
1	−89	41	−77	0	−13/40
1	−149	49	−125	−12	−81/280
1	−89	−59	−77	−72	−71/280
1	−39	11	−32	−5	−61/280
1	−22	−5	−19	−9	−51/280
1	−32	−5	−21	−19	−41/280
1	−69	29	−37	−32	−31/280
1	−79	29	−77	20	−3/40
1	−102	−5	−71	−49	−11/280
1	−22	3	−21	1	−1/280
1	−80	38	−79	29	9/280
1	−45	−22	−39	−29	19/280
1	−42	0	−29	−19	29/280
1	−12	3	−11	1	39/280
1	−51	11	−50	8	7/40
1	−39	19	−37	8	59/280
1	−142	68	−89	−51	69/280
1	−10	3	−9	1	79/280
1	−99	41	−85	−2	89/280
1	−42	8	−39	1	99/280
1	−51	21	−37	−12	109/280
1	−79	−41	−62	−60	17/40
1	−32	−2	−19	−19	129/280
1	−92	43	−79	−1	139/280

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Table 7, continued from previous page

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = 1$, $p_1 = -2$, $s_2 = 1/15$

sum	k_0	k_1	l_0	l_1	s_1
−1	52	−15	41	9	−139/280
−1	81	−31	57	22	−129/280
−1	82	−20	69	9	−17/40
−1	90	−28	61	29	−109/280
−1	60	2	41	29	−99/280
−1	152	−18	149	−11	−89/280
−1	57	−18	29	29	−79/280
−1	109	1	100	17	−69/280
−1	40	−18	29	9	−59/280
−1	72	25	69	29	−7/40
−1	132	10	89	69	−39/280
−1	100	−38	69	29	−29/280
−1	105	−8	89	21	−19/280
−1	141	−71	137	−40	−9/280
−1	130	−63	129	−51	1/280
−1	65	−28	61	−11	11/280
−1	42	−15	41	−11	3/40
−1	49	1	32	25	31/280
−1	65	2	61	9	41/280
−1	21	−11	17	2	51/280
−1	137	32	109	69	61/280
−1	89	−31	57	32	71/280
−1	230	−103	121	109	81/280
−1	70	−23	69	−19	13/40
−1	29	−11	17	12	101/280
−1	61	−31	57	−10	111/280
−1	141	−51	97	42	121/280
−1	57	32	49	41	131/280

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = 1$, $p_1 = -2$, $s_2 = 3/20$

sum	k_0	k_1	l_0	l_1	s_1
0	−127	21	−88	−46	−139/280
0	−68	34	−47	−19	−129/280
0	−119	53	−118	44	−17/40
0	−139	66	−126	12	−109/280
0	−76	2	−67	−14	−99/280
0	−56	22	−54	13	−89/280
0	−56	4	−49	−9	−79/280
0	−188	64	−177	26	−69/280
0	−38	4	−27	−14	−59/280
0	−22	1	−18	−6	−7/40
0	−199	66	−148	−46	−39/280
0	−48	−6	−39	−19	−29/280
0	−29	3	−16	−16	−19/280
0	−118	14	−99	−22	−9/280
0	−102	31	−96	12	1/280
0	−87	26	−86	22	11/280
0	−137	26	−128	4	3/40
0	−119	18	−108	−6	31/280
0	−22	−9	−16	−16	41/280
0	−59	26	−48	−6	51/280
0	−106	2	−67	−54	61/280
0	−196	92	−134	−57	71/280
0	−159	−19	−118	−76	81/280
0	−116	4	−77	−54	13/40
0	−47	21	−46	14	101/280
0	−182	21	−136	−58	111/280
0	−102	11	−56	−56	121/280
0	−66	24	−54	−7	131/280

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = 1$, $p_1 = -2$, $s_2 = 7/30$

sum	k_0	k_1	l_0	l_1	s_1
1	−69	16	−37	−35	−139/280
1	−34	11	−27	−5	−129/280
1	−47	15	−46	11	−17/40
1	−55	23	−29	−26	−109/280
1	−57	25	−54	11	−99/280
1	−29	−4	−27	−7	−89/280
1	−57	5	−34	−29	−79/280
1	−64	−26	−55	−37	−69/280
1	−37	13	−29	−6	−59/280
1	−69	34	−65	13	−7/40
1	−24	6	−17	−7	−39/280
1	−86	36	−77	5	−29/280
1	−14	4	−7	−7	−19/280
1	−35	13	−21	−14	−9/280
1	−16	6	−15	3	1/280
1	−84	26	−57	−27	11/280
1	−25	13	−24	6	3/40
1	−75	−7	−56	−34	31/280
1	−61	−4	−45	−27	41/280
1	−27	13	−21	−4	51/280
1	−125	33	−109	−6	61/280
1	−54	−29	−47	−37	71/280
1	−109	44	−107	33	81/280
1	−29	14	−27	5	13/40
1	−65	33	−54	−4	101/280
1	−87	33	−61	−24	111/280
1	−127	45	−94	−29	121/280
1	−34	16	−25	−7	131/280

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = 1$, $p_1 = -2$, $s_2 = 19/60$

sum	k_0	k_1	l_0	l_1	s_1
−1	97	−45	96	−36	−139/280
−1	27	7	24	11	−129/280
−1	75	−23	51	24	−17/40
−1	85	−33	59	24	−109/280
−1	44	−16	27	17	−99/280
−1	117	45	106	59	−89/280
−1	91	14	57	57	−79/280
−1	15	−3	11	4	−69/280
−1	47	15	44	19	−59/280
−1	76	−16	75	−13	−7/40
−1	26	−6	25	−3	−39/280
−1	91	−36	85	−13	−29/280
−1	116	−16	87	35	−19/280
−1	174	−74	97	77	−9/280
−1	25	−13	19	4	1/280
−1	19	4	17	7	11/280
−1	107	−43	84	16	3/40
−1	44	16	35	27	31/280
−1	74	−6	45	37	41/280
−1	87	5	59	44	51/280
−1	104	−36	67	37	61/280
−1	96	−36	67	27	71/280
−1	34	−14	17	17	81/280
−1	75	47	64	59	13/40
−1	36	−16	27	7	101/280
−1	51	−16	27	25	111/280
−1	84	−21	67	15	121/280
−1	105	−43	59	46	131/280

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Table 7, continued from previous page

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = 1$, $p_1 = -2$, $s_2 = 2/5$

sum	k_0	k_1	l_0	l_1	s_1
0	−104	36	−91	−1	−139/280
0	−188	49	−184	36	−129/280
0	−72	−4	−68	−11	−17/40
0	−184	48	−171	12	−109/280
0	−111	27	−104	8	−99/280
0	−52	16	−51	12	−89/280
0	−76	−13	−52	−44	−79/280
0	−44	−24	−36	−33	−69/280
0	−84	−52	−76	−61	−59/280
0	−204	16	−181	−28	−7/40
0	−192	76	−156	−21	−39/280
0	−196	47	−164	−24	−29/280
0	−24	8	−23	4	−19/280
0	−161	37	−144	−4	−9/280
0	−192	56	−173	4	1/280
0	−133	−31	−124	−44	11/280
0	−204	88	−196	47	3/40
0	−333	9	−232	−144	31/280
0	−112	36	−101	4	41/280
0	−84	−32	−81	−36	51/280
0	−51	12	−44	−4	61/280
0	−264	−32	−183	−141	71/280
0	−244	76	−141	−108	81/280
0	−124	28	−116	7	13/40
0	−304	136	−221	−71	101/280
0	−148	29	−92	−64	111/280
0	−164	48	−161	37	121/280
0	−144	16	−141	9	131/280

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = 1$, $p_1 = -2$, $s_2 = 29/60$

sum	k_0	k_1	l_0	l_1	s_1
1	−39	−1	−32	−12	−139/280
1	−72	28	−51	−19	−129/280
1	−80	−12	−51	−49	−17/40
1	−49	19	−27	−22	−109/280
1	−20	8	−11	−9	−99/280
1	−81	11	−75	−2	−89/280
1	−111	51	−107	28	−79/280
1	−41	21	−40	13	−69/280
1	−115	48	−89	−19	−59/280
1	−12	5	−11	1	−7/40
1	−59	1	−40	−27	−39/280
1	−71	−9	−52	−35	−29/280
1	−119	49	−112	20	−19/280
1	−160	68	−89	−71	−9/280
1	−39	−11	−32	−20	1/280
1	−90	38	−89	31	11/280
1	−40	13	−39	9	3/40
1	−27	−12	−21	−19	31/280
1	−31	11	−27	0	41/280
1	−109	41	−75	−32	51/280
1	−39	−19	−32	−27	61/280
1	−79	19	−75	8	71/280
1	−107	−22	−69	−69	81/280
1	−67	−40	−59	−49	13/40
1	−99	39	−92	13	101/280
1	−49	21	−35	−12	111/280
1	−91	41	−72	−12	121/280
1	−99	−59	−92	−67	131/280

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = 2$, $p_1 = 2$, $s_2 = 0/1$

sum	k_0	k_1	l_0	l_1	s_1
0	−33	12	−32	8	−33/70
0	−124	−13	−112	−32	−61/140
0	−152	−52	−109	−103	−2/5
0	−112	4	−68	−59	−51/140
0	−52	−12	−44	−23	−23/70
0	−148	71	−132	8	−41/140
0	−193	92	−132	−56	−9/35
0	−79	−28	−76	−32	−31/140
0	−68	31	−52	−12	−13/70
0	−56	8	−53	1	−3/20
0	−116	−52	−89	−83	−4/35
0	−12	4	−9	−3	−11/140
0	−123	−33	−92	−72	−3/70
0	−233	76	−172	−56	−1/140
0	−172	84	−133	−28	1/35
0	−164	7	−152	−16	9/140
0	−52	24	−43	−4	1/10
0	−112	4	−103	−13	19/140
0	−152	68	−108	−39	6/35
0	−93	36	−72	−16	29/140
0	−212	68	−188	1	17/70
0	−173	52	−172	48	39/140
0	−96	28	−84	−3	11/35
0	−92	44	−83	7	7/20
0	−213	36	−116	−112	27/70
0	−112	48	−108	27	59/140
0	−139	47	−116	−12	16/35
0	−133	−28	−116	−52	69/140

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = 2$, $p_1 = 2$, $s_2 = 1/12$

sum	k_0	k_1	l_0	l_1	s_1
1	−77	29	−75	20	−33/70
1	−124	40	−111	3	−61/140
1	−17	−1	−10	−10	−2/5
1	−70	−30	−61	−41	−51/140
1	−91	9	−75	−20	−23/70
1	−19	0	−11	−11	−41/140
1	−44	5	−27	−21	−9/35
1	−59	10	−41	−21	−31/140
1	−151	69	−140	21	−13/70
1	−141	69	−140	56	−3/20
1	−54	5	−41	−17	−4/35
1	−21	9	−20	5	−11/140
1	−27	−11	−20	−19	−3/70
1	−61	29	−60	21	−1/140
1	−157	−1	−150	−14	1/35
1	−91	3	−75	−24	9/140
1	−75	20	−61	−11	1/10
1	−101	43	−90	5	19/140
1	−57	9	−40	−20	6/35
1	−141	49	−139	40	29/140
1	−51	23	−44	0	17/70
1	−107	−1	−64	−60	39/140
1	−51	−1	−35	−24	11/35
1	−80	21	−51	−31	7/20
1	−41	9	−35	−4	27/70
1	−37	19	−34	5	59/140
1	−40	−4	−31	−17	16/35
1	−131	53	−100	−24	69/140

continued next page

Table 7, continued from previous page

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = 2$, $p_1 = 2$, $s_2 = 1/6$

sum	k_0	k_1	l_0	l_1	s_1
−1	60	5	41	31	−33/70
−1	34	0	27	11	−61/140
−1	85	−16	61	27	−2/5
−1	61	−13	60	−10	−51/140
−1	51	21	40	34	−23/70
−1	69	−20	51	17	−41/140
−1	31	−13	24	5	−9/35
−1	31	−3	20	14	−31/140
−1	11	−3	10	0	−13/70
−1	41	−9	40	−6	−3/20
−1	7	1	5	4	−4/35
−1	51	−13	40	10	−11/140
−1	85	−26	81	−13	−3/70
−1	87	−29	74	5	−1/140
−1	85	0	51	47	1/35
−1	37	11	30	20	9/140
−1	74	5	51	37	1/10
−1	31	7	29	10	19/140
−1	50	−11	31	21	6/35
−1	71	−13	45	30	29/140
−1	97	−9	85	14	17/70
−1	45	0	41	7	39/140
−1	69	−30	67	−19	11/35
−1	125	−56	121	−33	7/20
−1	60	14	47	31	27/70
−1	29	−10	21	7	59/140
−1	45	−6	31	17	16/35
−1	47	−19	45	−10	69/140

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = 2$, $p_1 = 2$, $s_2 = 1/4$

sum	k_0	k_1	l_0	l_1	s_1
0	−138	66	−137	54	−33/70
0	−128	46	−97	−26	−61/140
0	−58	26	−57	19	−2/5
0	−44	−8	−42	−11	−51/140
0	−4	2	−2	−1	−23/70
0	−106	43	−78	−24	−41/140
0	−88	32	−47	−42	−9/35
0	−77	14	−44	−38	−31/140
0	−108	42	−82	−21	−13/70
0	−48	12	−47	9	−3/20
0	−18	6	−17	3	−4/35
0	−44	22	−37	−2	−11/140
0	−91	−42	−68	−68	−3/70
0	−28	−4	−26	−7	−1/140
0	−42	−11	−28	−28	1/35
0	−98	−8	−61	−57	9/140
0	−118	26	−117	23	1/10
0	−108	12	−87	−26	19/140
0	−34	2	−21	−17	6/35
0	−108	36	−67	−42	29/140
0	−68	−14	−66	−17	17/70
0	−204	42	−197	23	39/140
0	−122	9	−68	−68	11/35
0	−157	54	−118	−34	7/20
0	−82	3	−58	−34	27/70
0	−124	62	−117	23	59/140
0	−101	38	−78	−18	16/35
0	−117	−2	−108	−18	69/140

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = 2$, $p_1 = 2$, $s_2 = 1/3$

sum	k_0	k_1	l_0	l_1	s_1
1	−125	−5	−102	−41	−33/70
1	−89	−25	−81	−36	−61/140
1	−105	51	−97	14	−2/5
1	−96	39	−65	−29	−51/140
1	−105	35	−92	−1	−23/70
1	−85	−5	−81	−12	−41/140
1	−169	−5	−102	−96	−9/35
1	−161	4	−145	−25	−31/140
1	−125	−5	−81	−66	−13/70
1	−126	−41	−89	−85	−3/20
1	−36	−17	−29	−25	−4/35
1	−129	35	−106	−17	−11/140
1	−185	91	−162	4	−3/70
1	−145	55	−137	24	−1/140
1	−169	15	−161	−2	1/35
1	−86	39	−85	31	9/140
1	−85	11	−82	4	1/10
1	−46	23	−45	15	19/140
1	−146	63	−145	55	6/35
1	−145	−45	−137	−56	29/140
1	−81	38	−65	−9	17/70
1	−76	14	−65	−9	39/140
1	−69	−25	−56	−41	11/35
1	−161	68	−145	11	7/20
1	−12	−1	−9	−5	27/70
1	−65	−5	−42	−36	59/140
1	−65	11	−57	−6	16/35
1	−89	15	−82	−1	69/140

Smooth structures on a homeomorphism class of Eschenburg-Kruggel spaces with $|r| = 5$, $s = 2$, $p_1 = 2$, $s_2 = 5/12$

sum	k_0	k_1	l_0	l_1	s_1
−1	65	19	46	42	−33/70
−1	126	−49	85	39	−61/140
−1	25	5	16	16	−2/5
−1	42	−9	25	19	−51/140
−1	111	12	85	49	−23/70
−1	146	−58	119	15	−41/140
−1	169	−45	126	42	−9/35
−1	31	−8	25	5	−31/140
−1	35	−1	31	6	−13/70
−1	86	−9	55	39	−3/20
−1	129	−35	96	32	−4/35
−1	52	−4	45	9	−11/140
−1	86	42	75	55	−3/70
−1	49	−15	36	12	−1/140
−1	9	−5	6	2	1/35
−1	79	−25	66	7	9/140
−1	39	5	26	22	1/10
−1	26	6	19	15	19/140
−1	105	5	76	47	6/35
−1	127	6	109	35	29/140
−1	156	−44	125	25	17/70
−1	26	−9	25	−5	39/140
−1	95	−45	62	31	11/35
−1	49	−25	46	−9	7/20
−1	96	−33	69	25	27/70
−1	31	−14	29	−5	59/140
−1	89	5	72	31	16/35
−1	55	−11	47	6	69/140

5.4. **Appendix D.** The graphs in figure 2 depict the outcome of some time trials that compared alternative methods of computing the invariants of the Eschenburg-Kruggel spaces. The `C++` code is significantly faster. The sole exception to this statement is when $\max |k| \leq 4$. The reason for this apparent anomaly is that the `C++` code search and found the Eschenburg-Kruggel spaces in addition to computing their invariants; the Maple code merely computed the invariants.

The time trials were conducted on a quad-core Intel Xeon 5148 with an over-clocked 2.33GHz cpu, a 4MB cache, 3.2GB ram memory, and 8.4GB swap memory.

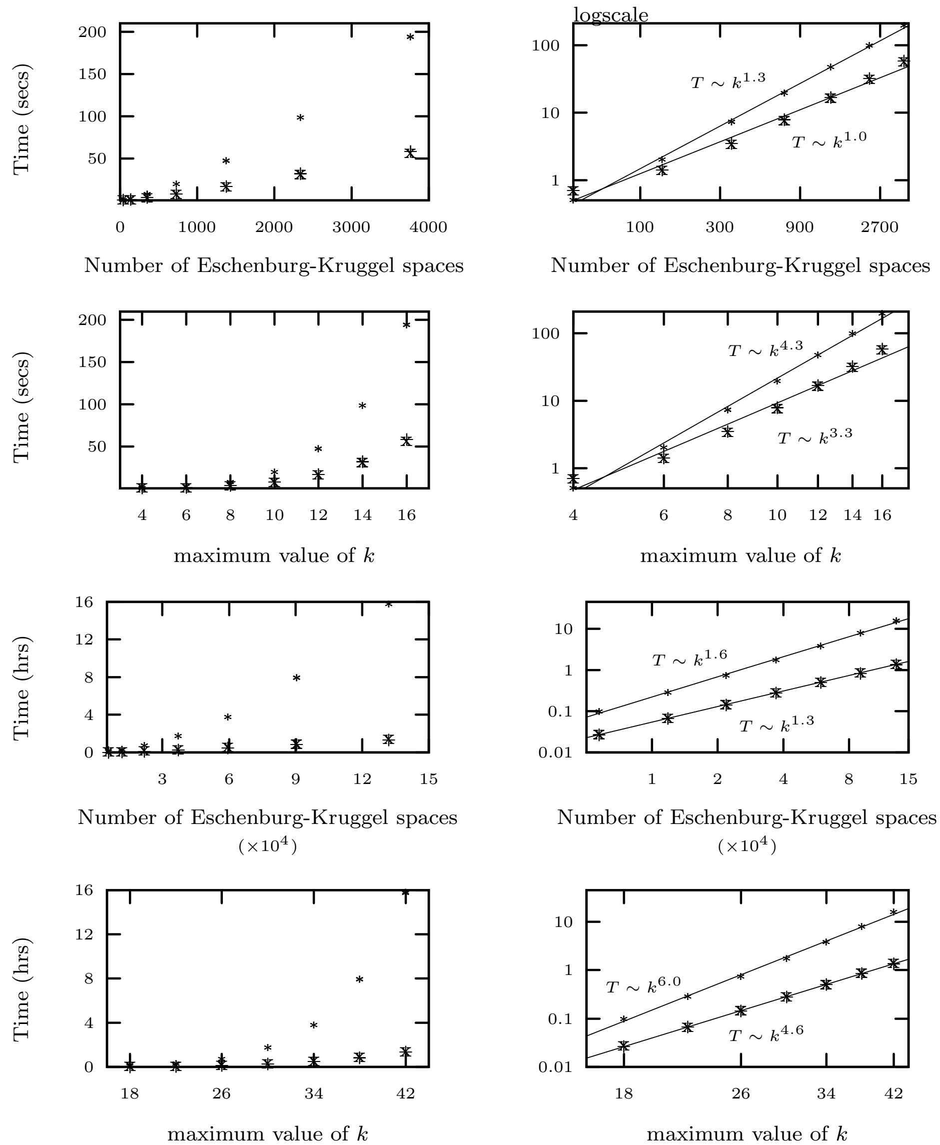

Figure 2: Even rows: Computation time versus the number, $N = N(k)$, of Eschenburg-Kruggel spaces in the cube $[-k, k]^6$ with $\sum_i k_i = \sum_i l_i \in [0, 2]$; Odd rows: Computation time versus the number k. The right column shows the same data in log-scale. $*$=Maple [11] times; $+$, x=C++-times.

References

[1] S. Aloff and N. Wallach. An infinite family of 7-manifolds admitting positively curved Riemannian structures. *Bull. Am. Math. Soc.* 81:93–97. 1975.

[2] L. Astey,. E. . Micha and G. Pastor. Homeomorphism and diffeomorphism types of Eschenburg spaces. *Differential Geom. Appl.* 7(1):41–50. 1997.

[3] L. T. Butler. The Maslov cocycle, smooth structures and real-analytic complete integrability. *Am. J. Math.* To appear.

[4] T. Chinburg, C. Escher and W. Ziller. Topological properties of Eschenburg spaces and 3-Sasakian manifolds. *Math. Ann.* 339(1): 3–20. 2007.

[5] J. H. Eschenburg. New examples of manifolds with strictly positive curvature. *Inv. Math.* 66:469–480. 1982.
[6] The GMP team. GNU MP *The Gnu Multiple Precision Arithmetic Library*. Edition 4.2.2. Free Software Foundation, Inc. 2007. `http://gmplib.org/`
[7] M. Kreck and S. Stolz. Some non diffeomorphic homeomorphic homogeneous 7-manifolds with positive sectional curvature. *J. Diff. Geom.* 33: 465–486. 1991.
[8] M. Kreck and S. Stolz. A diffeomorphism classification of 7-dimensional homogeneous Einstein manifolds with $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$-symmetry. *Ann. of Math. (2)* 127(2):373–388. 1988.
[9] B. Kruggel Kreck–Stolz invariants, normal invariants and the homotopy classification of generalized Wallach spaces. *Quart. J. Math. Oxford Ser. (2)* 49:469–485. 1998.
[10] B. Kruggel Homeomorphism and diffeomorphism classification of Eschenburg spaces. *Quart. J. Math. Oxford Ser. (2)*. 56:553–577. 2005.
[11] Maplesoft, Inc. *Maple, a computer algebra system.* Version 11. 2008. `http://www.maplesoft.com/`.
[12] Maxima.sourceforge.net. *Maxima, a computer algebra system.* Version 5.17.1. 2008. `http://maxima.sourceforge.net/`.
[13] P. Nelson. *BC - An arbitrary precision calculator language.* Version 1.06.94. 2008. `http://www.gnu.org/software/bc/`.
[14] J. Wilkening. Gnu Multiple Precision Arithmetic Frontend for `C++`. October 2008. `http://math.berkeley.edu/~wilken/code/gmpfrxx/`

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